Exercise Set Solutions #1

"Discrete Mathematics" (2025)

E1. How many symmetric $n \times n$ matrices are there with entries chosen from the numbers $[q]$?

> ***Solution:*** A symmetric $n \times n$ matrices is defined by $\frac{n(n+1)}{2}$ elements. Each element can be chosen in q different ways. Therefore, there are $q^{\frac{n(n+1)}{2}}$ symmetric $n \times n$ matrices.

E2. Let $n \in \mathbb{N}$ be even. How many permutations does the set $[n]$ have such that

(a) the sum of the first two elements is odd?

> ***Solution:*** Since n is an even integer, there are $\frac{n}{2}$ even and $\frac{n}{2}$ odd numbers in the set $[n]$. The first entry can be chosen in n different ways. If the value of the first entry is even, we have to choose an odd number for the second entry to make sure that the sum of the first two elements is odd. Analogously, if the first entry is odd, then we have to choose an even number for the second entry. Therefore, there are $\frac{n}{2}$ different ways to choose the second entry. For the remaining $n-2$ entries we have no restriction. Due to the previous observations, there are $\frac{n^2}{2}(n-2)\,!$ permutions of the set $[n]$ such that the first two elements sum up to an odd number.

(b) the last two elements sum up to n.

> ***Solution:*** If the last element would be n, then the second last element would have to be 0 , but 0 is not an element in $[n]$. Furthermore, if the last element would be $n/2$ then the second last element would have to be $n/2$ as well. This is not possible, because in $[n]$ each number appears just once. Therefore, there are $n-2$ different ways to choose the last element. Let m be the value of the last element, then the second last element is $n-m$. For the remaining entries 1 to $n-2$ there are no restrictions. Hence the number of permutations of the set $[n]$ such that the last two elements sum up to n is $(n-2)(n-2)!$.

E3. In how many different ways can the letters of the word MATHEMATICALLY be arranged such that

(a) the word starts always with MA?

> ***Solution:*** Since the words have to start with MA, the first two letters are fixed. If the remaining letters in THEMATICALLY would all be different then there would be 12! different ways to arrange the arrange the word MATHEMATICALLY such that all words start with MA. Since THEMATICALLY contains twice the letters A,T,L, each of which multiplies the counting by 2!. Hence, the solution is $\frac{12!}{2!\cdot 2!\cdot 2!}$.

(b) the three A's are adjacent?

> ***Solution:*** In order to obtain words such that the As are adjacent, we consider all permutations of the letters in MATHEMTICLLY (we removed all As except of one) and add afterwards the two missing As to the single A. The number of permutations of the letters in MATHEMTICLLY is $12!/(2! \cdot 2! \cdot 2!)$ because it contains twice the letters $\mathrm{M}, \mathrm{T}, \mathrm{L}$.

(c) the word MATH is always included?

Solution: Note that MATH can occur in any rearrangement of MATHEMATICALLY at most once because it contains H .

In order to obtain the number of different ways one can arrange the letters in MATHEMATICALLY such that all words contain MATH, we remove the letters MATH and instead replace it by a new letter Θ (this is a greek letter, so it is outside A-Z). Then, we consider the number of ways we can arrange ΘEMATICALLY. This can be done in $\frac{11!}{2!\cdot 2!}$ ways.

(d) the word MAT is included twice?

Solution: In how many ways can you arrange the letters of MATHEMATICALLY such that MAT is always included twice? Replace each occurence of MAT with Θ. Then the problem is simply asking the number of ways we can rearrange $\Theta E\Theta ICALLY$. This can be done in $\frac{10!}{2!\cdot 2!}$ ways.

E4. In how many different ways can we distribute 40 bottles of water to 10 kids and 5 adults, such that each kid gets at least one bottle?

Since each kid has to get at least one bottle of water, we give every kid one bottle and then we have to count the number of ways to distribute 30 bottles of water to 15 people. By applying Proposition 1.14 of the lecture notes, the solution is

$$\binom{30+15-1}{15-1} = \binom{44}{14}$$

E5. Let $n, r, k \in \mathbb{N}$ and $n \geq r \geq k$. Prove the following two equations:

(a)

$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}.$$

Solution: We give two different ways to prove

$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$$

By applying $\binom{n}{k} = \frac{n!}{(n-k)!k!}$, we get

$$\binom{n}{k}\binom{n-k}{r-k} = \frac{n!}{(n-k)!k!} \cdot \frac{(n-k)!}{(n-r)!(r-k)!} = \frac{n!}{(n-r)!(r-k)!k!}.$$

Furthermore,

$$\binom{n}{r}\binom{r}{k} = \frac{n!}{(n-r)!r!} \cdot \frac{r!}{(r-k)!k!} = \frac{n!}{(n-r)!(r-k)!k!}$$

We have shown that

$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}.$$

Next, we prove the equation by using a combintaorial argument: Let N be a set of n elements. Let $A, B \subseteq N$ with $|A| = k$ and $|B| = r - k$. Then $\binom{n}{k}$ gives the number of ways one can choose A from N. In order to obtain B there are $n-k$ elements left and therefore there are $\binom{n-k}{r-k}$ different ways. Hence $\binom{n}{k}\binom{n-k}{r-k}$ gives the number of ways to obtain A and B from N.

Let $C \subseteq N$ with $|C| = r$. We set $B = C \backslash A$. In order to obtain A and B, we first compute the number of different ways to obtain C from N and then we compute the number of ways to obtain A from C. Therefore, $\binom{n}{r}\binom{r}{k}$ is the number of ways to obtain A and B from N. We have shown that the number of ways to obtain A and B from N is equal to $\binom{n}{k}\binom{n-k}{r-k}$ and equal to $\binom{n}{r}\binom{r}{k}$. This implies $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$.

(b)

$$\binom{n}{k} = \sum_{j=k-1}^{n-1} \binom{j}{k-1}.$$

Solution: We want to prove $\binom{n}{k} = \sum_{j=k}^{n} \binom{j-1}{k-1}$. Let us assume we have a set $N = [n]$ and $A \subseteq N$ where $|A| = k$ and j is the maximal element in A. This means $j \in A$ and there are $k-1$ elements of A with value less than j. Therefore, the number of different ways to choose A is $\binom{j-1}{k-1}$. To determine the number of ways to obtain k elements of $[n]$, the value of j can be anything between a and n. By taking the sum $\sum_{j=k}^{n} \binom{j-1}{k-1}$ we determine the number of different ways to obtain k elements of $[n]$. We already know that the number of ways to obtain k elements of $[n]$ is $\binom{n}{k}$, thus $\binom{n}{k} = \sum_{j=k}^{n} \binom{j-1}{k-1}$.

As a second way of proving this equality we can make use of the additive relation between the binomial coefficients from Proposition 1.11, i.e.

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Using this identity repeatedly we obtain

$$\begin{aligned}
\binom{n}{k} &= \binom{n-1}{k-1} + \binom{n-1}{k} \\
&= \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-2}{k} \\
&= \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-3}{k-1} + \binom{n-3}{k} \\
&\vdots \\
&= \left(\sum_{j=k}^{n-1} \binom{j}{k-1} \right) + \binom{k}{k}
\end{aligned}$$

Since $\binom{k}{k} = 1 = \binom{k-1}{k-1}$, the last sum equals $\sum_{j=k-1}^{n-1} \binom{j}{k-1}$, which proves the claim.